



Chern–Simons formulation of three-dimensional gravity with torsion and nonmetricity

Sergio L. Cacciatori^{a,c}, Marco M. Caldarelli^{b,c}, Alex Giacomini^{b,c},
Dietmar Klemm^{b,c,*}, Diego S. Mansi^{b,c}

^a *Dipartimento di Matematica dell'Università di Milano, Via Saldini 50, I-20133 Milano, Italy*

^b *Dipartimento di Fisica dell'Università di Milano, Via Celoria 16, I-20133 Milano, Italy*

^c *INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy*

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Dedicated to the memory of Antonio Meucci

Abstract

We consider various models of three-dimensional gravity with torsion or nonmetricity (metric affine gravity), and show that they can be written as Chern–Simons theories with suitable gauge groups. Using the groups $ISO(2, 1)$, $SL(2, \mathbb{C})$ and $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, and the fact that they admit two independent coupling constants, we obtain the Mielke–Baekler model for zero, positive and negative effective cosmological constant respectively. Choosing $SO(3, 2)$ as the gauge group, one gets a generalization of conformal gravity that has zero torsion and only the trace part of the nonmetricity. This characterizes a Weyl structure. Finally, we present a new topological model of metric affine gravity in three dimensions arising from an $SL(4, \mathbb{R})$ Chern–Simons theory.

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* Corresponding author at: Dipartimento di Fisica dell'Università di Milano, Via Celoria 16, I-20133 Milano, Italy.

E-mail addresses: cacciatori@mi.infn.it (S.L. Cacciatori), marco.caldarelli@mi.infn.it (M.M. Caldarelli), alex.giacomini@mi.infn.it (A. Giacomini), dietmar.klemm@mi.infn.it (D. Klemm), diego.mansi@mi.infn.it (D.S. Mansi).

1. Introduction

General relativity in four spacetime dimensions is a notoriously difficult theory, already at the classical and in particular at the quantum level. This is one of the main reasons why people are interested in simpler models that nevertheless retain almost all of the essential features of four-dimensional general relativity. One such model is pure gravity in $2 + 1$ dimensions, with or without the cosmological constant. This theory has been studied extensively in the past, in particular by Deser, Jackiw and 't Hooft [1,2]. The most famous example where we learned something on general relativity by considering a simpler toy model is perhaps the BTZ black hole [3], whose study revealed a lot on the quantum structure and the statistical mechanics of black holes (for a review cf. [4]).

Major progress in $(2 + 1)$ -dimensional gravity came when Achúcarro and Townsend [5] and Witten [6] showed that these systems can be written as Chern–Simons (CS) theories, with gauge group $ISO(2, 1)$, $SL(2, \mathbb{C})$ or $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ for zero, positive or negative cosmological constant respectively. In trying to write down a CS action for the Poincaré group, one encounters the problem that $ISO(2, 1)$ is not semisimple, and therefore the Killing form is degenerate. As noted by Witten [6], the Poincaré algebra admits nevertheless a nondegenerate, ad-invariant bilinear form given by

$$\langle J_a, P_b \rangle = \eta_{ab}, \quad \langle J_a, J_b \rangle = \lambda \eta_{ab}, \quad \langle P_a, P_b \rangle = 0, \tag{1.1}$$

where J_a and P_a denote the Lorentz and translation generators, and λ is an arbitrary real constant. Mathematically, the existence of an ad-invariant, nondegenerate quadratic form on the Poincaré algebra follows from the fact that $iso(2, 1)$ is the double extension of a reductive Lie algebra (in this case the trivial algebra): Let \mathcal{A} be a reductive Lie algebra, i.e., a direct sum of a semisimple and an abelian algebra. \mathcal{A} admits an invariant nondegenerate bilinear form Ω_{ij} , whose restriction to the semisimple part is simply given by the Killing form, and the restriction to the abelian subalgebra is proportional to the identity. The generators τ_i of \mathcal{A} satisfy

$$[\tau_i, \tau_j] = f_{ij}{}^k \tau_k.$$

The double extension of \mathcal{A} is obtained by adding the new generators H_a and H_b^* such that

$$\begin{aligned} [\tau_i, \tau_j] &= f_{ij}{}^k \tau_k + h_{ij}{}^{\bar{a}} H_{\bar{a}}^*, \\ [H_a, \tau_i] &= h_{ai}{}^j \tau_j, \\ [H_a, H_b] &= g_{ab}{}^c H_c, \\ [H_a, H_b^*] &= g_{\bar{a}\bar{b}}{}^{\bar{c}} H_{\bar{c}}^*, \\ [\tau_i, H_{\bar{a}}^*] &= [H_{\bar{a}}^*, H_b^*] = 0, \end{aligned} \tag{1.2}$$

where $h_{ai}{}^j \Omega_{jk} = h_{ik}{}^{\bar{a}} \delta_{\bar{a}a}$. If furthermore $g_{ab}{}^c = g_{\bar{a}\bar{b}}{}^{\bar{c}}$, there exists an ad-invariant, nondegenerate quadratic form on the double extension of \mathcal{A} , given by [7]

$$\Omega_{IJ} = \begin{pmatrix} \Omega_{ij} & 0 & 0 \\ 0 & \lambda_{ab} & \delta_{\bar{a}\bar{b}} \\ 0 & \delta_{\bar{a}\bar{b}} & 0 \end{pmatrix}, \tag{1.3}$$

where $I = i, a, \bar{a}$, and λ_{ab} denotes any invariant quadratic form on the algebra generated by the H_a . If the algebra \mathcal{A} is trivial (no generators τ_i), (1.2) has exactly the structure of the Poincaré

algebra

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \quad [J_a, P_b] = \epsilon_{ab}{}^c P_c, \quad [P_a, P_b] = 0, \quad (1.4)$$

if we identify the generators H_a with J_a and H_b^* with P_b . The invariant quadratic form (1.3) reduces then to (1.1). Retaining a nonvanishing λ in a CS formulation of three-dimensional gravity leads to the inclusion of a gravitational Chern–Simons action (i.e., a CS term for the spin connection) [6]. This does not change the classical equations of motion, but leads to modifications at the quantum level [6].

One can now try to depart from pure gravity, thus rendering the model less trivial, while maintaining at the same time its integrability. A possible way to introduce additional structure is to permit nonvanishing torsion and/or nonmetricity (metric affine gravity) [8]. We would like to do this in such a way that the resulting model can still be written as a CS theory for some gauge group. There are several reasons that motivate the introduction of torsion or nonmetricity. Let us mention here just a few of them. For a more detailed account we refer the reader to [8]. First of all, nonmetricity is a measure for the violation of local Lorentz invariance [8], which has become fashionable during the last years. Second, the geometrical concepts of nonmetricity and torsion have applications in the theory of defects in crystals, where they are interpreted as densities of point defects and line defects (dislocations) respectively, cf. [9] and references therein. Finally, nonmetric connections or connections with torsion are interesting from a mathematical point of view. For example, a torsionless connection that has only the trace part of the nonmetricity characterizes a so-called Weyl structure. If, moreover, the symmetric part of the Ricci tensor is proportional to the metric, one has an Einstein–Weyl structure (cf. e.g. [10]). Einstein–Weyl manifolds represent the analogue of Einstein spaces in Weyl geometry, and are less trivial than the latter, which have necessarily constant curvature in three dimensions. Einstein–Weyl structures are interesting also due to their relationship to certain integrable systems, like the $SU(\infty)$ Toda [11] or the dispersionless Kadomtsev–Petviashvili equation [10].

In this paper, we consider various models of three-dimensional metric affine gravity and show that they can be written as CS theories. This is accomplished either by using gauge groups larger than $ISO(2, 1)$, $SL(2, \mathbb{C})$ or $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, or by using the fact that these groups admit two independent coupling constants, as was explained above for the case of the Poincaré group.

The remainder of our paper is organized as follows. In the next section, we briefly summarize the basic notions of metric affine gravity. In Section 3, we show that the Mielke–Baekler model, which is characterized by nonvanishing torsion and zero nonmetricity, can be written as a CS theory for arbitrary values of the effective cosmological constant. In Section 4, a CS action for the conformal group $SO(3, 2)$ is considered, and it is shown that this leads to a generalization of conformal gravity with a Weyl connection. Finally, in Section 5, we propose a topological model of metric affine gravity based on an $SL(4, \mathbb{R})$ CS theory and discuss some of its solutions. In the last section we summarize the results and draw some conclusions.

2. Metric affine gravity

In order to render this paper self-contained, we summarize briefly the basic notions of metric affine gravity. For a detailed review see [8].

The standard geometric set-up of Einstein’s general relativity is a differential manifold \mathcal{M} , of dimension D , endowed with a metric g and a Levi-Civita connection $\tilde{\nabla}$, which is uniquely determined by the requirements of metricity ($\tilde{\nabla}g = 0$) and vanishing torsion. This structure is known as a semi-Riemannian space (\mathcal{M}, g) .

One can now consider more complicated non-Riemannian geometries, where a new generic connection ∇ is introduced on $T\mathcal{M}$ which is, in general, independent of the metric. In this way one defines a new mathematical structure called a *metric affine space* (\mathcal{M}, g, ∇) .

One can measure the deviation from the standard geometric set-up by computing the difference $(\nabla - \tilde{\nabla})v$ between the action of the two connections on a vector field v defined on $T\mathcal{M}$. To be more specific one can choose a chart, so that the action of the connection is described by its coefficients,¹

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\lambda} v^\lambda, \tag{2.1}$$

and the deviation can be written as

$$(\nabla_\mu - \tilde{\nabla}_\mu)v^\nu = N^\nu_{\mu\lambda} v^\lambda. \tag{2.2}$$

The tensor

$$N^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu} \tag{2.3}$$

is called the *distortion* and measures the deviation of ∇ from the Levi-Civita connection. This object can be decomposed into different parts, depending essentially on two quantities: the *torsion* and the *nonmetricity*.

The torsion tensor T^a is defined by the first Cartan structure equation

$$T^a \equiv de^a + \omega^a_b \wedge e^b, \tag{2.4}$$

where ω^a_b is the spin connection acting on (flat) tangent space indices a, b, \dots , and e^a denotes the vielbein satisfying $e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}$, with η^{ab} the flat Minkowski metric. A priori, ω^a_b is independent of the connection coefficients $\Gamma^\lambda_{\mu\nu}$. The two objects become dependent on each other by the tetrad postulate

$$\nabla_\mu e^a_\nu = 0, \tag{2.5}$$

implying

$$\omega^a_{\mu b} = e^a_\lambda \Gamma^\lambda_{\mu\rho} e_b^\rho - e_b^\lambda \partial_\mu e^a_\lambda, \tag{2.6}$$

so that the spin connection $\omega^a_{\mu b}$ is the gauge transform of $\Gamma^\lambda_{\mu\rho}$ with transformation matrix e^a_λ .

For nonvanishing torsion, the connection coefficients are no longer symmetric in their lower indices, as can be seen from

$$0 = 2\nabla_{[\mu} e^a_{\nu]} = \left(\partial_\mu e^a_{\nu} - \partial_\nu e^a_{\mu} + \omega^a_{\mu b} e^b_{\nu} - \omega^a_{\nu b} e^b_{\mu} \right) - 2\Gamma^\lambda_{[\mu\nu]} e^a_{\lambda}, \tag{2.7}$$

which yields

$$T^\lambda_{\mu\nu} \equiv e_a^\lambda T^a_{\mu\nu} = 2\Gamma^\lambda_{[\mu\nu]}, \tag{2.8}$$

or, equivalently,

$$T^\lambda_{\mu\nu} = 2N^\lambda_{[\mu\nu]}, \tag{2.9}$$

since the Levi-Civita connection has zero torsion, $\tilde{\Gamma}^\lambda_{[\mu\nu]} = 0$.

¹ The connection coefficients for the Levi-Civita connection are called Christoffel symbols and are denoted by $\tilde{\Gamma}^\nu_{\mu\lambda}$.

The nonmetricity Q is a tensor which measures the failure of the metric to be covariantly constant,

$$Q_{\lambda\mu\nu} \equiv -\nabla_{\mu}g_{\nu\lambda}. \tag{2.10}$$

Using $\tilde{\nabla}g = 0$ and the definition (2.3), one gets

$$Q_{\lambda\mu\nu} = N_{\lambda\mu\nu} + N_{\nu\mu\lambda}, \tag{2.11}$$

where $N_{\lambda\mu\nu} = g_{\lambda\sigma}N^{\sigma}_{\mu\nu}$. For nonzero nonmetricity, the spin connection ω^{ab} is no longer antisymmetric in a, b : By computing the covariant derivative $\nabla_{\mu}\eta^{ab}$ one obtains $Q_{\mu}^{ab} = 2\omega_{\mu}^{(ab)}$, with $Q_{\mu}^{ab} \equiv e^{a\rho}e^{b\lambda}Q_{\rho\mu\lambda}$. This means that the spin connection takes values in $\mathfrak{gl}(D, \mathbb{R})$ instead of the Lorentz algebra $\mathfrak{so}(D - 1, 1)$.

Notice that in the presence of nonmetricity, the scalar product of two vectors u, v can change when u, v are transported parallel along a curve. Let t be the tangent vector of an infinitesimal curve c . The variation of the scalar product is then given by

$$\delta g(u, v) = \nabla_t(g_{\mu\nu}u^{\mu}v^{\nu}) = -Q_{\mu\lambda\nu}t^{\lambda}u^{\mu}v^{\nu}. \tag{2.12}$$

Physically, this states that if we enlarge the Lorentz group, the interval is no longer an invariant and in fact, for generic nonmetricity, the very concept of the light cone is lost.

The two tensors T and Q uniquely determine the distortion and, as a result, the connection. This can also be seen by counting the degrees of freedom: the distortion is a generic tensor with three indices, so it has D^3 independent components. The torsion and the nonmetricity, due to their symmetry properties, have respectively $D^2(D - 1)/2$ and $D^2(D + 1)/2$ independent components; their sum gives precisely the expected number of degrees of freedom. To obtain the distortion in terms of torsion and nonmetricity one has to solve the Eqs. (2.9) and (2.11). Considering all possible permutations one obtains

$$N_{\lambda\mu\nu} = \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}) + \frac{1}{2}(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu}), \tag{2.13}$$

which is the expected decomposition of the distortion. The Levi-Civita connection is obtained setting $T^a = 0$ and $Q_{ab} = 0$. The combination

$$K_{\nu\lambda\mu} = \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}), \tag{2.14}$$

which is antisymmetric in the first two indices, is also called contorsion.

Note that in metric affine gravity, the local symmetry group is the affine group $A(D, \mathbb{R}) \cong GL(D, \mathbb{R}) \times \mathbb{R}^D$ instead of the Poincaré group $ISO(D - 1, 1)$. The associated gauge fields are ω^{ab} and e^a . In what follows, we shall specialize to the case $D = 3$.

3. The Mielke–Baekler model as a Chern–Simons theory

Let us first consider the case of Riemann–Cartan spacetimes, characterized by vanishing nonmetricity, but nonzero torsion. A simple three-dimensional model that yields nonvanishing torsion was proposed by Mielke and Baekler (MB) [12] and further analyzed by Baekler, Mielke and Hehl [13]. The action reads [12]

$$I = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4, \tag{3.1}$$

where a, Λ, α_3 and α_4 are constants,

$$\begin{aligned}
 I_1 &= 2 \int e_a \wedge R^a, \\
 I_2 &= -\frac{1}{3} \int \epsilon_{abc} e^a \wedge e^b \wedge e^c, \\
 I_3 &= \int \omega_a \wedge d\omega^a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c, \\
 I_4 &= \int e_a \wedge T^a,
 \end{aligned}$$

and

$$\begin{aligned}
 R^a &= d\omega^a + \frac{1}{2} \epsilon^a{}_{bc} \omega^b \wedge \omega^c, \\
 T^a &= de^a + \epsilon^a{}_{bc} \omega^b \wedge e^c,
 \end{aligned}$$

denote the curvature and torsion two-forms respectively. ω^a is defined by $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{bc}$ with $\epsilon_{012} = 1$. I_1 yields the Einstein–Hilbert action, I_2 a cosmological constant, I_3 is a Chern–Simons term for the connection, and I_4 represents a translational Chern–Simons term. Note that, in order to obtain the topologically massive gravity of Deser, Jackiw and Templeton (DJT) [14] from (3.1), one has to add a Lagrange multiplier term that ensures vanishing torsion. The field equations following from (3.1) take the form

$$\begin{aligned}
 2aR^a - \Lambda \epsilon^a{}_{bc} e^b \wedge e^c + 2\alpha_4 T^a &= 0, \\
 2aT^a + 2\alpha_3 R^a + \alpha_4 \epsilon^a{}_{bc} e^b \wedge e^c &= 0.
 \end{aligned}$$

In what follows, we assume $\alpha_3\alpha_4 - a^2 \neq 0$.² Then the equations of motion can be rewritten as

$$2T^a = A \epsilon^a{}_{bc} e^b \wedge e^c, \quad 2R^a = B \epsilon^a{}_{bc} e^b \wedge e^c, \tag{3.2}$$

where

$$A = \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \quad B = -\frac{a \Lambda + \alpha_4^2}{\alpha_3 \alpha_4 - a^2}.$$

Thus, the field configurations are characterized by constant curvature and constant torsion. The curvature R^a of a Riemann–Cartan spacetime can be expressed in terms of its Riemannian part \tilde{R}^a and the contorsion one-form K^a by

$$R^a = \tilde{R}^a - dK^a - \epsilon^a{}_{bc} \omega^b \wedge K^c - \frac{1}{2} \epsilon^a{}_{bc} K^b \wedge K^c, \tag{3.3}$$

where $K^a_\mu = \frac{1}{2} \epsilon^a{}_{bc} e^{b\beta} e^{c\gamma} K_{\beta\gamma\mu}$, and $K_{\beta\gamma\mu}$ denotes the contorsion tensor given by (2.14). Using the equations of motion (3.2) in (3.3), one gets for the Riemannian part

$$2\tilde{R}^a = \Lambda_{\text{eff}} \epsilon^a{}_{bc} e^b \wedge e^c, \tag{3.4}$$

with the effective cosmological constant

$$\Lambda_{\text{eff}} = B - \frac{A^2}{4}.$$

² For $\alpha_3\alpha_4 - a^2 = 0$ the theory becomes singular [13].

This means that the metric is given by the (anti-)de Sitter or Minkowski solution, depending on whether Λ_{eff} is negative, positive or zero. It is interesting to note that Λ_{eff} can be nonvanishing even if the bare cosmological constant Λ is zero [13]. In this simple model, dark energy (i.e., Λ_{eff}) would then be generated by the translational Chern–Simons term I_4 .

In [15] it was shown that for $\Lambda_{\text{eff}} < 0$, the Mielke–Baekler model (3.1) can be written as a sum of two $\text{SL}(2, \mathbb{R})$ Chern–Simons theories. We will now show that this can be generalized to the case of arbitrary effective cosmological constant. For positive Λ_{eff} , the action I becomes a sum of two $\text{SL}(2, \mathbb{C})$ Chern–Simons theories with complex coupling constants, whereas for vanishing Λ_{eff} , I can be written as a CS theory for the Poincaré group.

To start with, we briefly summarize the results of [15]. For $\Lambda_{\text{eff}} < 0$ the geometry is locally AdS_3 , which has the isometry group $\text{SO}(2, 2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, so if the MB model is equivalent to a Chern–Simons theory, one expects a gauge group $\text{SO}(2, 2)$. Indeed, if one defines the $\text{SL}(2, \mathbb{R})$ connections

$$A^a = \omega^a + q e^a, \quad \tilde{A}^a = \omega^a + \tilde{q} e^a,$$

then the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern–Simons action³

$$I_{CS} = \frac{t}{8\pi} \int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle + \frac{\tilde{t}}{8\pi} \int \left\langle \tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right\rangle \quad (3.5)$$

coincides (up to boundary terms) with I in (3.1), if the parameters q, \tilde{q} and the coupling constants t, \tilde{t} are given by

$$q = -\frac{A}{2} + \sqrt{-\Lambda_{\text{eff}}}, \quad \tilde{q} = -\frac{A}{2} - \sqrt{-\Lambda_{\text{eff}}} \quad (3.6)$$

and

$$\frac{t}{2\pi} = 2\alpha_3 + \frac{2a + \alpha_3 A}{\sqrt{-\Lambda_{\text{eff}}}}, \quad \frac{\tilde{t}}{2\pi} = 2\alpha_3 - \frac{2a + \alpha_3 A}{\sqrt{-\Lambda_{\text{eff}}}}. \quad (3.7)$$

We see that q, \tilde{q} , and thus the connections A^a, \tilde{A}^a are real for negative Λ_{eff} . The coupling constants t, \tilde{t} are also real, but in general different from each other due to the presence of I_3 .

For $\Lambda_{\text{eff}} > 0$, q and \tilde{q} become complex, with $\tilde{q} = \bar{q}$ and thus $\tilde{A}^a = \bar{A}^a$. As the connections are no longer real, we must consider the complexification $\text{SL}(2, \mathbb{C})$ of $\text{SL}(2, \mathbb{R})$. Then (3.5) becomes a sum of two $\text{SL}(2, \mathbb{C})$ Chern–Simons actions, with complex coupling constants t, \tilde{t} , where $\tilde{t} = \bar{t}$. Again, (3.5) is equal (modulo boundary terms) to the Mielke–Baekler action (3.1). This of course makes sense, since the isometry group of three-dimensional de Sitter space is $\text{SO}(3, 1) \cong \text{SL}(2, \mathbb{C})$. The usual CS formulation of dS_3 gravity [6] is recovered for $\alpha_3 = \alpha_4 = 0$.

The real part of t , i.e., up to prefactors, α_3 , is subject to a topological quantization condition coming from the maximal compact subgroup $\text{SU}(2)$ of $\text{SL}(2, \mathbb{C})$ [16]. As $\tilde{t} = \bar{t}$, the action (3.5) leads to a unitary quantum field theory [16].

Finally, we come to the case of vanishing Λ_{eff} . The condition $B - A^2/4 = 0$ implies that Λ can be expressed in terms of the other parameters according to

$$\Lambda = \frac{2a^3 - 3a\alpha_3\alpha_4 \pm 2(a^2 - \alpha_3\alpha_4)^{\frac{3}{2}}}{\alpha_3^2}. \quad (3.8)$$

³ In (3.5), $\langle \tau_a, \tau_b \rangle = 2 \text{Tr}(\tau_a \tau_b) = \eta_{ab}$, and the $\text{SL}(2, \mathbb{R})$ generators τ_a satisfy $[\tau_a, \tau_b] = \epsilon_{ab}{}^c \tau_c$.

As we want Λ to be real, we assume $a^2 - \alpha_3\alpha_4 > 0$. Let us consider the CS action

$$I_{CS} = \frac{k}{4\pi} \int \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{3.9}$$

where A denotes an $\text{iso}(2, 1)$ valued connection, and the quadratic form on the Poincaré algebra is given by (1.1). According to what was said in the introduction, this (nondegenerate) bilinear form is ad-invariant for any value of the parameter λ . If we decompose the connection as

$$A = e^a P_a + (\omega^a + \gamma e^a) J_a, \tag{3.10}$$

then the CS action (3.9) coincides, up to boundary terms, with (3.1) (where now Λ is not independent, but determined by (3.8)), if the constants k , λ and γ are chosen as

$$\frac{k}{4\pi} = \mp \sqrt{a^2 - \alpha_3\alpha_4}, \quad \lambda = \mp \frac{\alpha_3}{\sqrt{a^2 - \alpha_3\alpha_4}}, \quad \gamma = \frac{a \pm \sqrt{a^2 - \alpha_3\alpha_4}}{\alpha_3}. \tag{3.11}$$

In conclusion, we have shown that the Mielke–Baekler model can be written as a Chern–Simons theory for any value of the effective cosmological constant Λ_{eff} , whose sign determines the gauge group. This was accomplished by a nonstandard decomposition of the CS connection in terms of the dreibein and the spin connection, and by using the fact that the gauge groups considered admit two independent coupling constants. As the CS connection is flat, and thus entirely determined by holonomies, there are no propagating local degrees of freedom; hence there cannot be any gravitons in the MB model, contrary to the claim in [13].

It would be interesting to study the asymptotic dynamics of the Mielke–Baekler model in the case $\Lambda_{\text{eff}} < 0$, where the spacetime is locally AdS_3 . According to the AdS/CFT correspondence [17], (3.1) should then be equivalent to a two-dimensional conformal field theory on the boundary of AdS_3 , where the bulk fields e^a and ω^a are sources for the CFT energy–momentum current and spin current respectively. It was claimed in [15] that in general the putative CFT has two different central charges. (Unlike for the case $\alpha_3 = \alpha_4 = 0$, $a = 1/16\pi G$, $\Lambda = -1/l^2$, where $c_L = c_R = 3l/2G$ [18].) It would be interesting to compute these central charges explicitly, and to see whether the entropy of the Riemann–Cartan black hole [19] (which represents a generalization of the BTZ black hole with torsion) can be reproduced by counting CFT states using the Cardy formula. Like in [20], one expects the action (3.5) to reduce to a sum of two chiral WZNW actions on the conformal spacetime boundary. For $\alpha_3 = \alpha_4 = 0$, these two chiral actions combine into a single nonchiral WZNW model [20]. As the two $\text{SL}(2, \mathbb{R})$ CS actions in (3.5) have different coupling constants, it might be that in the general case this is no longer true, and one is left with a sum of two chiral WZNW models that have different central charges. It remains to be seen how this reduction works in detail.

4. Weyl structures from Chern–Simons theory

In this section we will show how to get Weyl structures, which are characterized by torsion-free connections that involve only the trace part of the nonmetricity, from Chern–Simons theory. To start with, let us consider conformal gravity in three dimensions, defined by the action [14]

$$I = \int \left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3} \omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a \right). \tag{4.1}$$

Here, ω denotes an $\text{so}(2, 1)$ valued (and hence metric) connection, which is not a fundamental variable, but is considered as a function of the dreibein, as is required by vanishing torsion. Therefore, variation of (4.1) leads to third-order differential equations, namely [14]

$$C^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu\alpha\beta} \nabla_\alpha L^{\nu\beta} = 0, \quad (4.2)$$

where $L_{\mu\nu}$ denotes the Schouten tensor defined by

$$L_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}. \quad (4.3)$$

$C^{\mu\nu}$ is known as Cotton–York tensor.⁴ It has zero trace, reflecting the conformal invariance of (4.1). In three dimensions, the Cotton–York tensor takes the role of the Weyl tensor (which is identically zero in 3D). $C_{\mu\nu}$ vanishes if and only if spacetime is conformally flat [22]. Eq. (4.1) is sometimes called the gravitational Chern–Simons action. Its supersymmetric extension was obtained in [23]. The dimensional reduction of the action (4.1), studied in [24], has recently been shown to describe a subsector of BPS solutions to gauged supergravity in four dimensions [25].

Originally, the gravitational Chern–Simons action was introduced by Deser, Jackiw and Templeton in order to render three-dimensional Einstein gravity nontrivial: If one adds (4.1) to the Einstein–Hilbert action, the theory acquires a propagating, massive, spin 2 degree of freedom [14].

Horne and Witten showed that conformal gravity in three dimensions with action (4.1) can be written as a Chern–Simons theory for the conformal group $\text{SO}(3, 2)$ [26]. To this end, they decomposed the $\text{SO}(3, 2)$ connection A according to

$$A_\mu = e^a{}_\mu P_a - \frac{1}{2} \omega_\mu^{ab} J_{ab} + \lambda^a{}_\mu K_a + \phi_\mu D, \quad (4.4)$$

where P_a , J_{ab} , K_a , D denote respectively the generators of translations, Lorentz transformations, special conformal transformations and dilations. The Chern–Simons action for A leads then to the equations of motion [26]

$$de^a + \omega^a{}_b \wedge e^b - \phi \wedge e^a = 0, \quad (4.5)$$

$$d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} - e^a \wedge \lambda^b + e^b \wedge \lambda^a = 0, \quad (4.6)$$

$$d\lambda^a + \omega^a{}_b \wedge \lambda^b + \phi \wedge \lambda^a = 0, \quad (4.7)$$

$$d\phi + e^a \wedge \lambda_a = 0. \quad (4.8)$$

The generator of an infinitesimal gauge transformation is

$$u = \rho^a P_a - \frac{1}{2} \tau^{ab} J_{ab} + \sigma^a K_a + \gamma D.$$

The transformation law $\delta A = -du - [A, u]$ leads then to

⁴ $C_{\mu\rho\nu} = \nabla_\mu L_{\nu\rho} - \nabla_\rho L_{\nu\mu}$ is called the Cotton two-form. See [21] for a nice review.

$$\begin{aligned}
 \delta e^a &= -d\rho^a - \omega^a{}_b \rho^b + e^b \tau^a{}_b - e^a \gamma + \phi \rho^a, \\
 \delta \omega^{ab} &= -d\tau^{ab} - \omega^a{}_c \tau^{cb} + \omega^b{}_c \tau^{ca} + e^a \sigma^b - e^b \sigma^a + \lambda^a \rho^b - \lambda^b \rho^a, \\
 \delta \lambda^a &= -d\sigma^a - \omega^a{}_b \sigma^b + \lambda^b \tau^a{}_b + \lambda^a \gamma - \phi \sigma^a, \\
 \delta \phi &= -d\gamma - e^a \sigma_a + \lambda^a \rho_a.
 \end{aligned}
 \tag{4.9}$$

Horne and Witten noticed that when the vielbein $e^a{}_\mu$ is invertible, the σ^a gauge invariance is precisely sufficient to set $\phi = 0$. With the gauge choice $\phi = 0$, the equations of motion simplify considerably. Eq. (4.5) implies then that the torsion vanishes. If we define $\lambda_{\mu\nu} = e_{a\mu} \lambda^a{}_\nu$, Eq. (4.8) means that $\lambda_{\mu\nu}$ is symmetric, whereas (4.6) leads to

$$\lambda_{\mu\nu} = R_{\mu\nu} - \frac{R}{4} g_{\mu\nu} = L_{\mu\nu},
 \tag{4.10}$$

so that $\lambda_{\mu\nu}$ represents the Schouten tensor. It is interesting to note that in this context, the Schouten tensor, which physically corresponds to a curvature, is at the same time a connection, namely the gauge field of special conformal transformations. Eq. (4.6) is then precisely the expression for the Riemann curvature tensor in terms of the Schouten tensor,

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho} L_{\nu\sigma} + g_{\nu\sigma} L_{\mu\rho} - g_{\mu\sigma} L_{\nu\rho} - g_{\nu\rho} L_{\mu\sigma},$$

valid in three dimensions. Finally, one has to interpret (4.7). To this end, one defines the connection coefficients $\Gamma^\mu{}_{\nu\rho}$ by requiring (2.5), which implies

$$\Gamma^\mu{}_{\nu\rho} = e_a{}^\mu \omega_{\nu b}^a e^b{}_\rho + e_a{}^\mu \partial_\nu e^a{}_\rho.$$

Eq. (4.7) is then equivalent to

$$\nabla_\mu L_{\nu\rho} - \nabla_\rho L_{\nu\mu} = 0,
 \tag{4.11}$$

which coincides with the equation of motion (4.2) following from the action (4.1). In the gauge $\phi = 0$, the gauge theory of the conformal group with Chern–Simons action is therefore equivalent to conformal gravity.

We can now ask what happens if one does not set $\phi = 0$. In this case it is convenient to define a generalized connection $\hat{\omega}$ by

$$\hat{\omega}^{ab} = \omega^{ab} - \eta^{ab} \phi.
 \tag{4.12}$$

Note that $\hat{\omega}^{ab}$ is no longer antisymmetric, and hence does not take values in the Lorentz algebra $so(2, 1) \cong sl(2, \mathbb{R})$, but in $gl(2, \mathbb{R})$. Therefore, this connection is not metric, but it is torsionless due to Eq. (4.5),

$$de^a + \hat{\omega}^a{}_b \wedge e^b = 0.$$

This can be solved to give

$$\begin{aligned}
 \hat{\omega}_\mu^{ab} &= \frac{1}{2} e^{av} (\partial_\mu e^b{}_v - \partial_v e^b{}_\mu) + \frac{1}{2} e^{av} e^{b\lambda} (\partial_\lambda e^c{}_v - \partial_v e^c{}_\lambda) e_{c\mu}, \\
 &\quad - \frac{1}{2} e^{bv} (\partial_\mu e^a{}_v - \partial_v e^a{}_\mu) + e^{av} \phi_v e^b{}_\mu - e^{bv} \phi_v e^a{}_\mu - \eta^{ab} \phi_\mu.
 \end{aligned}
 \tag{4.13}$$

As before, we require $\nabla_\mu e^a{}_\nu = 0$, which yields

$$\begin{aligned} \Gamma^\mu{}_{\nu\rho} &= e_a^\mu \hat{\omega}_{\nu b}^a e^b{}_\rho + e_a^\mu \partial_\nu e^a{}_\rho \\ &= \tilde{\Gamma}^\mu{}_{\nu\rho} + \phi^\mu g_{\nu\rho} - \phi_\rho \delta^\mu{}_\nu - \phi_\nu \delta^\mu{}_\rho, \end{aligned} \tag{4.14}$$

where $\tilde{\Gamma}^\mu{}_{\nu\rho}$ denotes the Christoffel connection. Using (4.14), one obtains for the nonmetricity

$$\nabla_\mu g_{\nu\rho} = 2g_{\nu\rho} \phi_\mu. \tag{4.15}$$

This is precisely the definition of a Weyl connection. Mathematically, a Weyl structure on a manifold \mathcal{M} is defined by a pair $W = (g, \phi)$, where g and ϕ are a Riemannian metric and a one-form on \mathcal{M} , respectively. There exists then one and only one torsion-free connection ∇ , called the Weyl connection, such that (4.15) holds. Note that Eq. (4.15) expresses the compatibility of ∇ with the conformal class of g . It is invariant under Weyl transformations

$$g_{\mu\nu} \rightarrow e^{2\chi} g_{\mu\nu}, \quad \phi_\mu \rightarrow \phi_\mu + \partial_\mu \chi, \tag{4.16}$$

where $\chi \in C^\infty(\mathcal{M})$. Historically, the connection satisfying (4.15) was introduced by Weyl in 1919 in an attempt to unify general relativity with electromagnetism [27].

We still have to interpret Eqs. (4.6)–(4.8). Eq. (4.8) implies that the antisymmetric part of $\lambda_{\mu\nu}$ represents essentially the field strength of ϕ ,

$$\lambda_{[\mu\nu]} = -\frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \equiv -\frac{1}{2} f_{\mu\nu}. \tag{4.17}$$

Let us denote the curvature two-form of the (metric, but not torsionless) connection ω by \mathcal{R} , i.e., $\mathcal{R}^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$. Note that the curvature of $\hat{\omega}$ splits as

$$R^{ab} = d\hat{\omega}^{ab} + \hat{\omega}^a{}_c \wedge \hat{\omega}^{cb} = \mathcal{R}^{ab} - \eta^{ab} f.$$

Eq. (4.6) is then equivalent to

$$\mathcal{R}_{\mu\nu\rho\sigma} = g_{\mu\rho} \lambda_{\nu\sigma} + g_{\nu\sigma} \lambda_{\mu\rho} - g_{\mu\sigma} \lambda_{\nu\rho} - g_{\nu\rho} \lambda_{\mu\sigma}, \tag{4.18}$$

which yields

$$\lambda_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4} \mathcal{R}^\rho{}_\rho g_{\mu\nu},$$

with $\mathcal{R}_{\mu\nu}$ denoting the Ricci tensor. Thus, $\lambda_{\mu\nu}$ represents again the Schouten tensor, but the one constructed from the connection ω , which differs from the full Schouten tensor associated with $\hat{\omega}$ by a piece proportional to the field strength $f_{\mu\nu}$. Eq. (4.18) expresses just the fact that in three dimensions the curvature is determined by the Schouten tensor alone. Note that this is still true in the presence of torsion, cf. Appendix A. Notice also that $\mathcal{R}_{\mu\nu\rho\sigma}$ is antisymmetric in its first two indices by virtue of the metricity of ω , but that $\mathcal{R}_{\mu\nu\rho\sigma} \neq \mathcal{R}_{\rho\sigma\mu\nu}$, because ω has nonvanishing torsion. Therefore the Ricci tensor $\mathcal{R}_{\mu\nu}$ is in general not symmetric.

Finally, we come to Eq. (4.7). Using $\nabla_\mu e_{a\nu} = 2\phi_\mu e_{a\nu}$, one gets

$$\nabla_\mu \lambda_{\nu\rho} - \nabla_\rho \lambda_{\nu\mu} = 0. \tag{4.19}$$

As $\lambda_{\mu\nu}$ does not represent the full Schouten tensor constructed from the connection $\hat{\omega}$, this equation cannot be interpreted as the vanishing of the Cotton two-form associated with the Weyl connection. If one so wishes, one can express $\lambda_{\mu\nu}$ in terms of the full Schouten tensor and the field strength f , and use the Bianchi identity for f to rewrite (4.19) as

$$C_{\mu\rho\nu} = -\nabla_\nu f_{\mu\rho}, \tag{4.20}$$

where $C_{\mu\rho\nu}$ denotes the Cotton two-form constructed from the Weyl connection $\hat{\omega}$. For zero ‘electromagnetic’ field ϕ , (4.19) means that the spacetime is conformally flat. Eq. (4.20) resembles the relation

$$C_{\mu\rho\nu} = -\frac{1}{4}g_{\nu\rho}\nabla^\lambda f_{\lambda\mu} + \frac{1}{4}g_{\nu\mu}\nabla^\lambda f_{\lambda\rho} - \frac{3}{2}\nabla_\nu f_{\mu\rho}, \tag{4.21}$$

that holds for Einstein–Weyl spaces [28], i.e., manifolds with a Weyl connection for which the symmetric part of the Ricci tensor is proportional to the metric,

$$R_{(\mu\nu)} = \frac{R}{3}g_{\mu\nu}. \tag{4.22}$$

The meaning of (4.20) can be clarified using the expression [28]

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \nabla_\mu\phi_\nu + 2\nabla_\nu\phi_\mu - g_{\mu\nu}\phi_\lambda\phi^\lambda + \phi_\mu\phi_\nu + g_{\mu\nu}\nabla_\lambda\phi^\lambda \tag{4.23}$$

for the Ricci tensor $R_{\mu\nu}$ of the Weyl connection in terms of the Ricci tensor $\tilde{R}_{\mu\nu}$ of the Levi-Civita connection (not to be confused with $\mathcal{R}_{\mu\nu}$), the one-form ϕ and its derivatives. Plugging (4.23) into (4.20) yields

$$\tilde{\nabla}_\mu\tilde{L}_{\nu\rho} - \tilde{\nabla}_\rho\tilde{L}_{\nu\mu} = 0, \tag{4.24}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection and $\tilde{L}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{\tilde{R}}{4}g_{\mu\nu}$. Eq. (4.24) means that, in terms of Riemannian data, spacetime is conformally flat, so that we have a Weyl structure defined on a conformally flat manifold.

The above results can also be understood from the point of view of gauge transformations: It is clear that, at least for an invertible triad, our model must be gauge-equivalent to conformal gravity, i.e., to the theory with $\phi_\mu = 0$. This means that there must be gauge transformations that take any solution of our theory to a conformally flat metric. Under a general gauge transformation g the connection changes according to

$$A' = g^{-1}Ag + g^{-1}dg. \tag{4.25}$$

For a special conformal transformation, $g = \exp(-\sigma^a K_a)$, (4.25) leads to

$$\phi'_\mu = \phi_\mu - e^a{}_\mu\sigma_a.$$

This vanishes if we choose $\sigma_a = e_a{}^\mu\phi_\mu$, which is always possible for an invertible triad.

We conclude this section by noting that the similarity of (4.20) with Eq. (4.21), valid for Einstein–Weyl spaces, suggests that there might be a relationship between the Einstein–Weyl equations and the Chern–Simons action for the conformal group. Note in this context that it is not known whether the Einstein–Weyl equations follow from an action principle, but it was conjectured in [28] that if such an action exists, it might be related to gravitational Chern–Simons forms. It would be interesting to explore this direction further.

5. Metric affine gravity from $SL(4, \mathbb{R})$ Chern–Simons theory

In the preceding section we saw how to get Weyl structures from Chern–Simons theory. The Weyl connection is a particular case of a nonmetric connection, where only the trace part is present. One might ask whether it is possible to write a more general metric affine gravity model as a Chern–Simons theory. We will show in this section that this is indeed possible.

As was explained in Section 2, in metric affine gravity, the Poincaré group $ISO(2, 1)$ is replaced by the affine group $A(3, \mathbb{R}) \cong GL(3, \mathbb{R}) \ltimes \mathbb{R}^3$. In attempting to write a CS action for the affine group, one encounters the problem that the Lie algebra $\mathfrak{a}(3, \mathbb{R})$ is neither reductive nor the double extension of some reductive Lie algebra, and therefore it does not admit an ad-invariant, nondegenerate quadratic form.⁵ A way out of this is to embed the affine group in some slightly larger group that is semisimple. The most obvious thing one can do is to consider the group $SL(4, \mathbb{R})$, which contains $A(3, \mathbb{R})$. Let us denote the generators of $SL(4, \mathbb{R})$ by L_{AB} , $A = 0, \dots, 3$, satisfying $\eta^{AB}L_{AB} = 0$, with $(\eta^{AB}) = \text{diag}(-1, 1, 1, 1)$. They obey the commutation relations

$$[L_{AB}, L_{CD}] = \eta_{AD}L_{CB} - \eta_{CB}L_{AD}.$$

Now split the generators into L_{ab} , $L_{3a} \equiv P_a$ and $L_{a3} \equiv K_a$, where $a = 0, 1, 2$. In this way one obtains

$$\begin{aligned} [L_{ab}, L_{cd}] &= \eta_{ad}L_{cb} - \eta_{cb}L_{ad}, \\ [L_{ab}, P_c] &= \eta_{ac}P_b, \quad [L_{ab}, K_c] = -\eta_{bc}K_a, \\ [K_a, P_b] &= -L_{ab} - \eta^{cd}L_{cd}\eta_{ab}, \\ [P_a, P_b] &= [K_a, K_b] = 0. \end{aligned} \tag{5.1}$$

We see that L_{ab} and P_c generate the subgroup $A(3, \mathbb{R})$. The chosen decomposition corresponds to rewriting the algebra $\mathfrak{sl}(4, \mathbb{R})$ as the graded algebra $\mathfrak{a}^* = \mathbb{R}^3 \oplus \mathfrak{gl}(3, \mathbb{R}) \oplus \mathbb{R}^{3*}$. Although this seemingly looks like a generalization of the conformal algebra, with $\mathfrak{so}(2, 1)$ replaced by $\mathfrak{gl}(3, \mathbb{R})$, one cannot identify the K_a with the generators of special conformal transformations. In fact, $SL(4, \mathbb{R})$ does not contain the conformal group $SO(3, 2)$ as a subgroup.⁶

Since $SL(4, \mathbb{R})$ is simple, it possesses (up to normalization) a unique gauge-invariant bilinear form, given by

$$\begin{aligned} \langle L_{ab}, L_{cd} \rangle &= \eta_{ad}\eta_{bc} - \frac{1}{4}\eta_{ab}\eta_{cd}, \quad \langle P_a, K_b \rangle = \eta_{ab}, \\ \langle L_{ab}, P_c \rangle &= \langle L_{ab}, K_c \rangle = 0, \quad \langle P_a, P_b \rangle = \langle K_a, K_b \rangle = 0. \end{aligned} \tag{5.2}$$

Let us decompose the connection according to

$$A = \sigma^{ba}L_{ab} + e^aP_a + \lambda^aK_a, \tag{5.3}$$

where we wish to interpret σ^{ab} as a $\mathfrak{gl}(3, \mathbb{R})$ valued connection and e^a as the dreibein. The physical significance of λ^a will become clear later. The generator of infinitesimal gauge transformations is a Lie-algebra valued zero-form,

$$u = \tau^{ba}L_{ab} + \rho^aP_a + \zeta^aK_a.$$

The variation of the gauge field A under a gauge transformation generated by u is

$$\delta A = -du - [A, u].$$

⁵ In fact it is straightforward to show explicitly that any ad-invariant quadratic form on $\mathfrak{a}(3, \mathbb{R})$ is necessarily degenerate.

⁶ Cf. footnote 11 of [8].

This means that the component fields transform as

$$\delta\sigma^{ba} = -d\tau^{ba} - (\sigma^b{}_c \tau^{ca} - \tau^b{}_c \sigma^{ca}) - (e^b \zeta^a + e^c \zeta_c \eta^{ba}) + (\lambda^a \rho^b + \lambda^c \rho_c \eta^{ba}), \tag{5.4}$$

$$\delta e^a = -d\rho^a + \tau^a{}_b e^b - \sigma^a{}_b \rho^b, \tag{5.5}$$

$$\delta\lambda^a = -d\zeta^a - \lambda^b \tau_b{}^a + \zeta^b \sigma_b{}^a. \tag{5.6}$$

With (5.3), the Chern–Simons action becomes

$$I_{CS} = \int \left(\sigma^a{}_b \wedge d\sigma^b{}_a + \frac{2}{3} \sigma^a{}_b \wedge \sigma^b{}_c \wedge \sigma^c{}_a - \frac{1}{4} \sigma \wedge d\sigma + e_a \wedge d\lambda^a + \lambda_a \wedge de^a + 2\lambda_a \wedge \sigma^{ab} \wedge e_b \right), \tag{5.7}$$

with $\sigma \equiv \sigma^a{}_a$. The equations of motion following from (5.7) read

$$de^a + \sigma^a{}_b \wedge e^b = 0, \tag{5.8}$$

$$d\lambda^a - \sigma^a{}_b \wedge \lambda^b = 0, \tag{5.9}$$

$$d\sigma^{ab} + \sigma^a{}_c \wedge \sigma^{cb} = -e^a \wedge \lambda^b - e^c \wedge \lambda_c \eta^{ab}. \tag{5.10}$$

Eq. (5.8) means that the torsion vanishes, $T^a = 0$. Defining

$$\omega^{ab} \equiv \sigma^{[ab]}, \quad Q^{ab} \equiv 2\sigma^{(ab)},$$

one can use (5.8) and (2.13) to obtain

$$\Gamma^\rho{}_{\mu\nu} = \tilde{\Gamma}^\rho{}_{\mu\nu} + \frac{1}{2} (Q^\rho{}_{\mu\nu} + Q^\rho{}_{\nu\mu} - Q_\mu{}^\rho{}_\nu), \tag{5.11}$$

where $\tilde{\Gamma}^\rho{}_{\mu\nu}$ denotes the Levi-Civita connection and $Q^\rho{}_{\mu\nu}$ is given by

$$Q^\rho{}_{\mu\nu} = e_a{}^\rho Q_{\mu c}^a e^c{}_\nu.$$

One easily verifies that $Q_{\nu\rho\mu} = Q_{\mu\rho\nu}$, and thus $\Gamma^\rho{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu}$, which is of course a consequence of vanishing torsion. Using (5.11), one obtains for the covariant derivative of the metric,

$$\nabla_\mu g_{\nu\lambda} = -Q_{\lambda\mu\nu},$$

so that $Q_{\lambda\mu\nu}$ represents the nonmetricity. Note that we also have $\nabla_\mu e_{a\nu} = -e_{b\nu} Q_{\mu a}^b$. Using this and the definition $\lambda_{\mu\nu} = e_{a\mu} \lambda^a{}_\nu$, Eq. (5.9) is seen to be equivalent to

$$\nabla_\mu \lambda_{\alpha\nu} - \nabla_\nu \lambda_{\alpha\mu} = 0, \tag{5.12}$$

whose deeper meaning will become clear below.

We finally come to Eq. (5.10). The curvature two-form is defined by

$$R^{ab} = d\sigma^{ab} + \sigma^a{}_c \wedge \sigma^{cb}. \tag{5.13}$$

Notice that in metric affine gravity, both the antisymmetry in the first two indices, and the block symmetry $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ of the curvature tensor are lost. This is why one can define two different Ricci tensors $R_{\mu\nu}$ and $S_{\mu\nu}$ (cf. Appendix A). Eq. (5.10) yields

$$R_{\alpha\beta\mu\nu} = -g_{\alpha\mu} \lambda_{\beta\nu} + g_{\alpha\nu} \lambda_{\beta\mu} - g_{\alpha\beta} \lambda_{\mu\nu} + g_{\alpha\beta} \lambda_{\nu\mu}, \tag{5.14}$$

and thus

$$\begin{aligned} R_{\mu\nu} &\equiv R^\alpha{}_{\mu\alpha\nu} = \lambda_{\nu\mu} - 3\lambda_{\mu\nu}, \\ S_{\mu\nu} &\equiv R_{\mu\nu}{}^\alpha{}_\alpha = -\lambda_{\nu\mu} + 2\lambda_{\mu\nu} - g_{\mu\nu}\lambda^\rho{}_\rho, \end{aligned} \tag{5.15}$$

from which we get

$$\lambda_{\mu\nu} = -\left(R_{\mu\nu} - \frac{R}{4}g_{\mu\nu} + S_{\mu\nu} - \frac{S}{4}g_{\mu\nu}\right). \tag{5.16}$$

Therefore, $\lambda_{\mu\nu}$ represents the sum of the two Schouten tensors constructed from $R_{\mu\nu}$ and $S_{\mu\nu}$. Eq. (5.12) means then that the sum of the two Cotton two-forms that one can construct must vanish. Eq. (5.12) is thus a direct generalization of the field equation of conformal gravity. The antisymmetric part of (5.14) yields

$$R_{[\alpha\beta]\mu\nu} = \frac{1}{2}(-g_{\alpha\mu}\lambda_{\beta\nu} + g_{\beta\mu}\lambda_{\alpha\nu} + g_{\alpha\nu}\lambda_{\beta\mu} - g_{\beta\nu}\lambda_{\alpha\mu}), \tag{5.17}$$

which means that the antisymmetrized curvature is given in terms of the sum of the two Schouten tensors alone. This is in fact nothing else than the irreducible decomposition of $R_{[\alpha\beta]\mu\nu}$ under the Lorentz group (cf. Appendix A), which comes out here as a field equation.

We still have to interpret the symmetric part of (5.14),

$$\begin{aligned} R_{(\alpha\beta)\mu\nu} &= \frac{1}{2}(\nabla_\mu Q_{\alpha\nu\beta} - \nabla_\nu Q_{\alpha\mu\beta}) \\ &= -\frac{1}{2}(g_{\alpha\mu}\lambda_{\beta\nu} + g_{\beta\mu}\lambda_{\alpha\nu} - g_{\alpha\nu}\lambda_{\beta\mu} - g_{\beta\nu}\lambda_{\alpha\mu}) - g_{\alpha\beta}(\lambda_{\mu\nu} - \lambda_{\nu\mu}). \end{aligned} \tag{5.18}$$

In arbitrary dimension, the irreducible decomposition of $R_{(\alpha\beta)\mu\nu} \equiv Z_{\alpha\beta\mu\nu}$ under the Lorentz group involves five pieces ${}^{(i)}Z_{\alpha\beta\mu\nu}$, $i = 1, \dots, 5$, with ${}^{(2)}Z$ vanishing identically in three dimensions [8]. Comparing the remaining four pieces given in Appendix A with the decomposition (5.18) implied by the field equations, we get

$$\begin{aligned} {}^{(1)}Z_{\alpha\beta\mu\nu} &= 0, \\ \Delta_{\mu\nu} &= -\frac{5}{3}\lambda_{[\mu\nu]}, \\ \Xi_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\lambda^\rho{}_\rho - \frac{3}{2}\lambda_{(\mu\nu)}, \\ {}^{(4)}Z_{\alpha\beta\mu\nu} &= -\frac{8}{3}g_{\alpha\beta}\lambda_{[\mu\nu]}. \end{aligned} \tag{5.19}$$

A priori, the symmetric part of the curvature has 18 independent components in three dimensions, but the field equation (5.18) tells us that seven of them must vanish (${}^{(1)}Z = 0$), and that the remaining ones are determined completely by the antisymmetric and the symmetric traceless part of the Schouten tensor $\lambda_{\mu\nu}$, that determines also the antisymmetric part of the curvature. Eqs. (5.12) and (5.18) are the only remaining equations for the metric $g_{\mu\nu}$ and the nonmetricity $Q_{\lambda\mu\nu}$.

5.1. Simple solutions

A simple solution of these equations can be obtained by setting $Q_{\lambda\mu\nu} = 0$. Then the connection reduces to the Christoffel connection. Furthermore, $R_{\mu\nu} = S_{\mu\nu}$, and $\lambda_{\mu\nu}$ becomes symmetric. In this case, Eq. (5.18) is satisfied iff $\lambda_{\mu\nu} = \frac{1}{3}\lambda^\rho{}_\rho g_{\mu\nu}$, which implies

$$R_{\mu\nu} = \frac{R}{3}g_{\mu\nu}, \tag{5.20}$$

i.e., the manifold is Einstein. For vanishing nonmetricity, we recover therefore general relativity. Note that the cosmological constant appears here as an integration constant, and not as an input. Notice also that Eq. (5.12) is then identically satisfied, since in three dimensions every Einstein space is conformally flat, and thus the Cotton two-form vanishes. The fact that the cosmological constant is no longer an external input can be seen also from a group-theoretic point of view. The assumption of vanishing nonmetricity selects the Lorentz generators from the $\mathfrak{gl}(3, \mathbb{R})$ generators, therefore reducing $\mathfrak{gl}(3, \mathbb{R})$ to $\mathfrak{so}(2, 1)$. The equations of motion imply that λ^a is proportional to the dreibein,

$$\lambda^a = \frac{\lambda}{3}e^a, \tag{5.21}$$

with $\lambda \equiv \lambda^\rho{}_\rho$. Introducing the Lorentz generators $J_{ab} = -2L_{[ab]}$, the connection becomes

$$A = \frac{1}{2}\omega^{ab}J_{ab} + e^a \left(P_a + \frac{\lambda}{3}K_a \right), \tag{5.22}$$

so that the new translation generators are given by

$$\Pi_a = P_a + \frac{\lambda}{3}K_a. \tag{5.23}$$

The J_{ab} and Π_a obey the algebra

$$\begin{aligned} [J_{ab}, J_{cd}] &= \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\ [J_{ab}, \Pi_c] &= \eta_{bc}\Pi_a - \eta_{ac}\Pi_b, \\ [\Pi_a, \Pi_b] &= \frac{\lambda}{3}J_{ab}. \end{aligned} \tag{5.24}$$

Depending on the sign of λ this is the algebra $\mathfrak{so}(3, 1)$ ($\lambda < 0$), $\mathfrak{so}(2, 2)$ ($\lambda > 0$) or $\mathfrak{iso}(2, 1)$ ($\lambda = 0$), generated by the isometries of de Sitter, anti-de Sitter and Minkowski spacetimes respectively, and the cosmological constant is given by $\Lambda = -\lambda/3$. An interesting observation is that these solutions enjoy a duality symmetry exchanging e^a and λ^a and relating large and small cosmological constants. To be more precise, if we act with the discrete transformation $e^a \mapsto \lambda^a$, $\lambda^a \mapsto e^a$, we obtain a new Einstein space solving the model with a cosmological constant $1/\Lambda$.

As a slight generalization let us consider the case when the nonmetricity has only a trace part, i.e.

$$Q_{\lambda\mu\nu} = -2g_{\lambda\nu}\phi_\mu.$$

This leads to

$$\nabla_\mu g_{\nu\lambda} = 2g_{\nu\lambda}\phi_\mu, \tag{5.25}$$

so that ∇ is a Weyl connection. If we define $F = d\phi$, Eq. (5.18) yields

$$g_{\alpha\beta}F_{\mu\nu} = \frac{1}{2}(g_{\alpha\mu}\lambda_{\beta\nu} + g_{\beta\mu}\lambda_{\alpha\nu} - g_{\alpha\nu}\lambda_{\beta\mu} - g_{\beta\nu}\lambda_{\alpha\mu}) + g_{\alpha\beta}(\lambda_{\mu\nu} - \lambda_{\nu\mu}). \quad (5.26)$$

Contracting with $g^{\beta\mu}$ and taking the symmetric part, one obtains

$$\lambda_{(\alpha\nu)} = \frac{1}{3}\lambda g_{\alpha\nu}. \quad (5.27)$$

Using this in (5.15) we get

$$R_{(\mu\nu)} = S_{(\mu\nu)} = -\frac{2}{3}\lambda g_{\mu\nu}, \quad (5.28)$$

which means that we have an Einstein–Weyl structure (cf. e.g. [10]). The antisymmetric part yields

$$\lambda_{[\alpha\nu]} = \frac{2}{7}F_{\alpha\nu}. \quad (5.29)$$

Inserting (5.27) and (5.29) in (5.26) and contracting with $g^{\alpha\beta}$ leads to $F = 0$, so that ϕ is pure gauge, $\phi = d\chi$ locally. This pure gauge nonmetricity can be eliminated by conformally rescaling

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = e^{-2\chi}g_{\mu\nu}.$$

The new metric \hat{g} satisfies then the same equations as in the case $Q = 0$, i.e., it is an Einstein metric. g is thus conformally Einstein.

5.2. Partial gauge fixing

As was explained in Section 2, the symmetry group that is gauged in metric affine gravity is the affine group $A(3, \mathbb{R})$. On the other hand, our model has the larger symmetry group $SL(4, \mathbb{R})$. In order to interpret the Chern–Simons theory considered above as a model of metric affine gravity, we have to gauge fix the additional symmetries, in the same way in which Horne and Witten gauge fixed the special conformal symmetries of the $SO(3, 2)$ CS theory considered in the previous section. We first show that one can use the additional symmetries generated by the K_a to set the trace part of the connection σ^{ab} to zero. Eq. (5.4) yields for the variation of the trace part under a gauge transformation

$$\delta(\sigma^{ba}\eta_{ba}) = -d(\tau^{ba}\eta_{ba}) - 4\zeta^a e_a + 4\lambda^a \rho_a. \quad (5.30)$$

This shows that for an invertible triad, the ζ^a gauge invariance is precisely sufficient to set $\sigma^{ba}\eta_{ba} = 0$. Furthermore, Eq. (5.5) says that the triad is completely unchanged by a ζ^a gauge transformation, so $e^a{}_\mu$ remains invertible in this new gauge. The gauge transformations that preserve this gauge are given by the $A(3, \mathbb{R})$ generators τ^{ab} and ρ^a , but from (5.30) we see that we must compensate with a ζ^a transformation that is determined entirely by the τ^{ab} and the ρ^a according to

$$\zeta_c = -\frac{1}{4}e_c{}^\mu \partial_\mu (\tau^{ab}\eta_{ab}) + e_c{}^\mu \lambda^a{}_\mu \rho_a. \quad (5.31)$$

Note that $\sigma^a{}_a = 0$ implies $Q^{\nu}{}_{\mu\nu} = 0$, and thus by (5.18) $\lambda_{[\mu\nu]} = 0$, so the tensor λ is symmetric in this gauge.

Next we show that the symmetries of the gauge fixed model consist of diffeomorphisms and local $GL(3, \mathbb{R})$ transformations, as they should for metric affine gravity. If we set $\rho^a = \zeta^a = 0$ in (5.4)–(5.6), the τ^{ab} give a local $GL(3, \mathbb{R})$ transformation. Local diffeomorphisms are not apparent in the transformation laws. Under a diffeomorphism generated by $-v^\mu$, the fields should transform as

$$\begin{aligned} \tilde{\delta}e^a{}_\mu &= -v^\nu(\partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu) - \partial_\mu(v^\nu e^a{}_\nu), \\ \tilde{\delta}\sigma_\mu^{ab} &= -v^\nu(\partial_\nu\sigma_\mu^{ab} - \partial_\mu\sigma_\nu^{ab}) - \partial_\mu(v^\nu\sigma_\nu^{ab}), \\ \tilde{\delta}\lambda^a{}_\mu &= -v^\nu(\partial_\nu\lambda^a{}_\mu - \partial_\mu\lambda^a{}_\nu) - \partial_\mu(v^\nu\lambda^a{}_\nu). \end{aligned} \tag{5.32}$$

This should be a gauge transformation in our theory. If we make a gauge transformation with gauge parameters $\rho^a = v^\nu e^a{}_\nu$, $\tau^{ab} = v^\nu\sigma_\nu^{ab}$, $\zeta^a = e^{a\mu}\lambda^b{}_\mu\rho_b$ (as required by (5.31)), this differs from the diffeomorphism by

$$\begin{aligned} \tilde{\delta}e^a{}_\mu - \delta e^a{}_\mu &= -v^\nu(\partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu + \sigma_{\nu c}^a e^c{}_\mu - \sigma_{\mu c}^a e^c{}_\nu), \\ \tilde{\delta}\sigma_\mu^{ab} - \delta\sigma_\mu^{ab} &= -v^\nu \left(\partial_\nu\sigma_\mu^{ab} - \partial_\mu\sigma_\nu^{ab} - e^{b\rho}\lambda^d{}_\rho e_{d\nu} e^a{}_\mu - \lambda^d{}_\mu e_{d\nu}\eta^{ab} \right. \\ &\quad \left. + \lambda^b{}_\mu e^a{}_\nu + \lambda_{c\mu} e^c{}_\nu \eta^{ab} - \sigma_{\mu d}^a \sigma_\nu^{db} + \sigma_\mu^{db} \sigma_{\nu d}^a \right), \\ \tilde{\delta}\lambda^a{}_\mu - \delta\lambda^a{}_\mu &= -v^\nu(\partial_\nu\lambda^a{}_\mu - \partial_\mu\lambda^a{}_\nu + \sigma_\mu^{ba} e_b{}^\rho \lambda^c{}_\rho e_{c\nu} - \lambda_{d\mu} \sigma_\nu^{da}). \end{aligned} \tag{5.33}$$

These differences vanish when the equations of motion (5.8)–(5.10) hold, and when $\lambda_{\mu\nu} = \lambda_{\nu\mu}$, which is satisfied in the gauge $\sigma^a{}_a = 0$ that we use. Thus, diffeomorphisms are gauge transformations on shell.

6. Conclusions

It is possible to geometrically extend general relativity in several ways, by allowing torsion or nonmetricity in the theory. In this article, we focused on three spacetime dimensions and showed how to write such generalized gravitational models as Chern–Simons theories. Starting from the usual formulation of three-dimensional gravity and using a nonstandard decomposition of the Chern–Simons connection, we recovered the Mielke–Baekler model for arbitrary sign of the effective cosmological constant by playing with the independent coupling constants admitted by the gauge group. In this way, we realized explicitly three-dimensional gravity with torsion as a Chern–Simons theory. Then, we turned to torsionless but nonmetric gravitational models. The simplest example is obtained by allowing only the trace part of the nonmetricity. This is Weyl’s gravity, and we proved its equivalence with the $SO(3, 2)$ Chern–Simons theory describing conformal gravity. Finally, we obtained a gravitational theory with more general nonmetricity by embedding the affine group $A(3, \mathbb{R})$ in the special linear group $SL(4, \mathbb{R})$ and writing a Chern–Simons action for the latter. It would be interesting to see whether it is possible to obtain a gravitational theory incorporating both nonmetricity and torsion from a Chern–Simons theory.

These gravitational models in reduced dimensionality are interesting because their integrability allows one to investigate important theoretical questions linked to the gravitational force. For instance, we already mentioned the asymptotic dynamics of the MB model in the $\Lambda_{\text{eff}} < 0$ case, which deserves further analysis to identify the corresponding dual field theory. This, in turn, would give the opportunity to understand the statistical mechanics of the Riemann–Cartan black hole.

Another important issue in theories where torsion and/or nonmetricity are present is the coupling with external matter. This is particularly problematic with generic nonmetricity, since the concept of the light cone, and hence of causality, ceases to be invariant. However, in the Chern–Simons models of gravity under consideration, one can easily write down an invariant action for a particle propagating on the backgrounds that it generates as a Wess–Zumino functional [29],

$$S_p = \int d\tau \langle K, g^{-1} \partial_\tau g \rangle, \tag{6.1}$$

where K is a constant element of the algebra, encoding the geometric properties of the particle (mass, spin, etc.) and $g(\tau)$ is an orbit of the gauge group of the gravitational theory under consideration. This formulation has the advantage of providing straightforwardly a symplectic form for the Hamiltonian description of the theory, which can be used to quantize the particle in a coordinate independent way. The analysis of such systems would allow one to define new intrinsic properties of the particles, analogous to mass and spin, but corresponding to the additional generators of the gauge group. Hopefully, this could provide some insight into the interpretation of metric affine theories, even in higher dimensional cases.

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Appendix A. Decomposition of curvature in metric affine gravity

In this section we briefly summarize the irreducible decomposition of the curvature under the Lorentz group, given in [8]. Thereby, we specialize to three dimensions.

Let us first consider the antisymmetric part of the curvature. One easily shows that

$$\epsilon_\gamma^{\alpha\beta} R_{\alpha\beta\mu\nu} \epsilon^{\mu\nu\rho} = 2R_{\rho\gamma} - R_{g\rho\gamma} + 2S_{\rho\gamma} - S_{g\rho\gamma}, \tag{A.1}$$

where

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad S_{\mu\nu} = R^\alpha_{\mu\nu\alpha}, \quad R = R^\mu_{\mu}, \quad S = S^\mu_{\mu} = R. \tag{A.2}$$

Note that in metric affine gravity, the Riemann tensor is no longer symmetric in the first two indices, so that one can define two different Ricci tensors $R_{\mu\nu}$ and $S_{\mu\nu}$, that in general are not symmetric. (For vanishing nonmetricity, but nonzero torsion, one has $R_{(\alpha\beta)\mu\nu} = 0$, so that the two Ricci tensors coincide. However, since $R_{\alpha\beta\mu\nu} \neq R_{\mu\nu\alpha\beta}$, the Ricci tensor is not symmetric.) Contracting (A.1) with $\epsilon_{\lambda\sigma}{}^\gamma \epsilon^{\rho}{}_{\tau\eta}$ yields

$$R_{[\sigma\lambda]\eta\tau} = \frac{1}{2}(g_{\sigma\eta}L_{\lambda\tau} + g_{\lambda\tau}L_{\sigma\eta} - g_{\sigma\tau}L_{\lambda\eta} - g_{\lambda\eta}L_{\sigma\tau}), \tag{A.3}$$

where $L_{\mu\nu}$ denotes the sum of the two Schouten tensors built from $R_{\mu\nu}$ and $S_{\mu\nu}$,

$$L_{\mu\nu} = R_{\mu\nu} - \frac{R}{4}g_{\mu\nu} + S_{\mu\nu} - \frac{S}{4}g_{\mu\nu}. \tag{A.4}$$

One can of course further decompose (A.3) into three pieces corresponding to the antisymmetric, symmetric trace-free, and trace parts of the sum of the two Ricci tensors [8].

In order to decompose the symmetric part $R_{(\alpha\beta)\mu\nu} \equiv Z_{\alpha\beta\mu\nu}$ of the curvature, one first splits Z into a traceless and a trace part,

$$Z_{\alpha\beta\mu\nu} = z_{\alpha\beta\mu\nu} + \frac{1}{3}g_{\alpha\beta}Z^\gamma{}_{\gamma\mu\nu}.$$

Then one gets [8]

$$Z_{\alpha\beta\mu\nu} = {}^{(1)}Z_{\alpha\beta\mu\nu} + {}^{(2)}Z_{\alpha\beta\mu\nu} + {}^{(3)}Z_{\alpha\beta\mu\nu} + {}^{(4)}Z_{\alpha\beta\mu\nu} + {}^{(5)}Z_{\alpha\beta\mu\nu}, \quad (\text{A.5})$$

with ${}^{(2)}Z$ identically vanishing in three dimensions and

$$\begin{aligned} {}^{(3)}Z_{\alpha\beta\mu\nu} &= \frac{3}{10}(g_{\alpha\mu}\Delta_{\beta\nu} - g_{\alpha\nu}\Delta_{\beta\mu} + g_{\beta\mu}\Delta_{\alpha\nu} - g_{\beta\nu}\Delta_{\alpha\mu}) - \frac{2}{5}g_{\alpha\beta}\Delta_{\mu\nu}, \\ {}^{(4)}Z_{\alpha\beta\mu\nu} &= \frac{1}{3}g_{\alpha\beta}Z^\gamma{}_{\gamma\mu\nu}, \\ {}^{(5)}Z_{\alpha\beta\mu\nu} &= \frac{1}{3}(g_{\alpha\mu}\Xi_{\beta\nu} - g_{\alpha\nu}\Xi_{\beta\mu} + g_{\beta\mu}\Xi_{\alpha\nu} - g_{\beta\nu}\Xi_{\alpha\mu}), \\ {}^{(1)}Z_{\alpha\beta\mu\nu} &= Z_{\alpha\beta\mu\nu} - {}^{(3)}Z_{\alpha\beta\mu\nu} - {}^{(4)}Z_{\alpha\beta\mu\nu} - {}^{(5)}Z_{\alpha\beta\mu\nu}, \end{aligned} \quad (\text{A.6})$$

where

$$\Delta_{\mu\nu} = z_\mu{}^\alpha{}_{\alpha\nu} - z_\nu{}^\alpha{}_{\alpha\mu}, \quad \Xi_{\mu\nu} = \frac{1}{2}(z_\mu{}^\alpha{}_{\alpha\nu} + z_\nu{}^\alpha{}_{\alpha\mu}).$$

In three dimensions, $Z_{\alpha\beta\mu\nu}$ has 18 independent components, and (A.5) corresponds to the decomposition $18 = 7 + 0 + 3 + 3 + 5$.

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